# Longitudinal Momentum in Flat Beam Transformer

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This note provides an expanded discussion relating to the use of the longitudinal momentum as it is at entry to the flat beam transformer for use in determining the transformer quadrupole parameters.

The two main assumptions or approximations that I used in writing and talking in 2001-2 were:

1. Particles are emitted from the cathode with negligible kinematic momentum into an ideal solenoidal magnetic field having only a longitudinal component of field, $B_0$

2. Other than the solenoidal field at the cathode, no other beamline element (such as a quadrupole) before the transformer will influence the angular momentum of the particles.

These assumptions were made primarily to make it easy to discuss the process. The zero-emiittance from the cathode didn't seem totally irrelevant in the context of TESLA linear collider numbers then current; I doubt that such is the case today for the ILC parameters. The kinematic momentum is defined by the familiar $\vec{p} = \gamma m \vec{v}$. At the end of this note there will be use of the canonical momentum, $\mathcal{P}$, which is related to the kinematic momentum by $p = \vec{\mathcal{P}} - \mathrm{e}\vec{A}$ where $\vec{A}$ is the vector potential.

Here is a sketch of a particle departing from the solenoid.

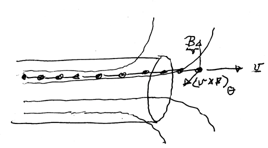


Figure 1: Solenoid and a particle exiting from it.

Imagine a right-handed coordinate system with the positive z-axis directed downstream. A particle displaced by $x$ will receive momentum in the $y$ direction of amount

$$p_y = \mathrm{e} \int (\vec{v} \times \vec{B})_y(x) dz = -\mathrm{e} v_z \int B_x dz. \tag{1}$$

In the second form of Eq. 1, the $v_x B_y$ term in the cross product has been ignored, even though some (small) $v_x$ will be developed while passing through the end field of the solenoid. The integral can be evaluated by Gauss's Law applied to a cylinder of radius $x$ stretching through the end region of the solenoid:

$$\pi x^2 B_0 = 2\pi x \int B_x dz. \tag{2}$$

The left hand side of Eq. 2 represents the flux entering the cylinder upstream and the right hand side states that all the flux departs through the cylindrical wall with none remaining to exit the downstream end. Combining Eqs. 1 and 2 gives

$$p_y = -\frac{\mathrm{e}B_0}{2} x, \tag{3}$$

and the same argument for a displacement in $y$ yields

$$p_x = \frac{\mathrm{e}B_0}{2} y. \tag{4}$$

If one follows a standard geometrical optics convention with $x' \equiv p_x/p_z$ and $y' \equiv p_y/p_z$ then

$$x' = ky, \quad y' = -kx \tag{5}$$

with

$$k \equiv \frac{B_0}{2p_z/\mathrm{e}} = \frac{B_0}{2(B\rho)}. \tag{6}$$

Eqs. 5 and 6 may be looked on as the initial conditions at exit from the solenoid. With the definitions

$$X \equiv \begin{pmatrix} x \\ x' \end{pmatrix}, \quad Y \equiv \begin{pmatrix} y \\ y' \end{pmatrix}, \quad S_0 \equiv \begin{pmatrix} 0 & \frac{1}{k} \\ -k & 0 \end{pmatrix} \tag{7}$$

this initial condition may be written

$$Y_0 = S_0 X_0 \tag{8}$$

Propagation forward to the upstream end of the transformer is represented by the same matrix $M$ in both transverse degrees of freedom, according to our assumptions. From $Y = MY_0 = MS_0X_0 = MS_0M^{-1}MX_0$, the matrix S at entry to the transformer is

$$S = MS_0M^{-1} \tag{9}$$

I started talking about $S$, giving it the name "correlation" matrix, in the Summer of 2001. The thin lens skew quadrupole settings were expressed in terms of $S$ at entry to the transformer. Daniel Mihalcea and Eric Thrane carried out simulations; there is some discussion in the LINAC2002 paper, a copy of which is attached. The algebraic relationships between $S$ and the skew quadrupole settings are also given there.

Although I was unsuccessful in doing much analytically with $S$, simple cases are easy. Suppose a thin ideal accelerating cavity is at the end of the solenoid and is followed by a drift of length $L$ leading to the transformer. Then

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_0}{p} \end{pmatrix} = \begin{pmatrix} 1 & \frac{p_0}{p} L \\ 0 & \frac{p_0}{p} \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 1 & -L \\ 0 & \frac{p}{p_0} \end{pmatrix} \tag{10}$$

where $p$ and $p_0$ are the $z$-components of momentum at entry to the transformer and exit from the solenoid respectively. Then combining Eqs. 9 and 10, $S$ at the transformer is given by

$$S = \begin{pmatrix} -k\frac{p_0}{p} L & \frac{1}{k}\frac{p}{p_0} + k\frac{p_0}{p} L^2 \\ -k\frac{p_0}{p} & k\frac{p_0}{p} L \end{pmatrix}. \tag{11}$$

The parameter $k$ is in effect multiplied by the momentum ratio to arrive at a new value of the parameter at the transformer.

In going through the end of the solenoid, use was made of the impulse approximation. An alternative way of arriving at the results is provided by use of canonical coordinates. With the assumption that the kinematic momentum is zero at the cathode, $\vec{\mathcal{P}} = \mathrm{e}\vec{A}$. For the ideal solenoid field, $\vec{A} = (B_0/2)x\vec{i} - (B_0/2)y\vec{j}$, where $\vec{i}$ and $\vec{j}$ are the usual unit vectors. So we have

$$\mathcal{P}_x = \frac{\mathrm{e}B_0}{2}, \qquad \mathcal{P}_y = -\frac{\mathrm{e}B_0}{2}. \tag{12}$$

Since these components of the canonical momenta are conserved through the end of the solenoid, we arrive again at Eqs. 4 and 5.